hep-th/0402005
SNB/February/2004

Nilpotent Symmetries For Matter Fields In Non-Abelian Gauge Theory: Augmented Superfield Formalism

R.P.Malik
S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake, Calcutta-700 098, India
E-mail address: malik@boson.bose.res.in

Abstract: In the framework of superfield approach to Becchi-Rouet-Stora-Tyutin (BRST) formalism, the derivation of the (anti-)BRST nilpotent symmetry transformations for the matter fields, present in any arbitrary interacting gauge theory, has been a long-standing problem. In our present investigation, the local, covariant, continuous and off-shell nilpotent (anti-)BRST symmetry transformations for the Dirac fields $(\psi, \bar{\psi})$ are derived in the framework of the augmented superfield formulation where the four $(3+1)$-dimensional (4D) interacting non-Abelian gauge theory is considered on the six $(4+2)$-dimensional supermanifold parametrized by the four *even* spacetime coordinates x^μ and a couple of *odd* elements (θ and $\bar{\theta}$) of the Grassmann algebra. The requirement of the invariance of the matter (super)currents and the horizontality condition on the (super)manifolds leads to the derivation of the nilpotent symmetries for the matter fields as well as the gauge- and the (anti-)ghost fields of the theory in the general scheme of augmented superfield formalism.

Keywords: Augmented superfield formulation; (anti-)BRST symmetries; horizontality condition; invariance of the matter (super)currents; 4D non-Abelian interacting gauge theory

PACS Nos.: 11.15.-q; 12.20.-m; 03.70.+k

1 Introduction

The symmetry groups and corresponding algebras have played some notable roles in the developments of the modern theoretical high energy physics up to the energy scale of the order of Planck's scale. In particular, the local and continuous symmetry groups corresponding to the gauge theories * have been found to dictate three (out of four) fundamental interactions of nature which govern physics up to the energy scale of the order of grand unification. The requirement of the *local* gauge invariance (i.e. gauge principle) enforces the existence of an interaction term in the Lagrangian density of the theory where the gauge field couples to the matter conserved current of the interacting gauge theory [3]. The existence of the conserved matter current owes its origin to the presence of a global gauge invariance in the theory (Noether's theorem). One of the most elegant and intuitive methods of covariantly quantizing such a class of gauge theories is the Becchi-Rouet-Stora-Tyutin (BRST) formalism where the unitarity and the "quantum" gauge (i.e. BRST) invariance are respected together at any arbitrary order of perturbative computations for a given physical process. The reach and range of the usefulness of this intuitive formalism have been beautifully extended to include the second-class constraints in its domain of applicability [4]. This formalism has now been found to hold a firm ground in the realm of the modern developments in the context of topological field theories [5-7], topological string theories [8], (super)string theories and their close cousins D-branes and M-theory (see, e.g., [9,10] and references therein). Its deep connection with the mathematics (see, e.g., [11-15] for details) of cohomology and differential geometry, its very striking similarity with some of the key concepts of supersymmetry, its natural inclusion in the Batalin-Vilkovisky scheme [16,17], its intuitive interpretation in the language of geometry on the supermanifold, etc., have been responsible for this active topic of research to soar a fairly high degree of mathematical sophistication and very useful (as well as attractive) physical applications.

In our present investigation, we shall be concentrating on the geometrical aspects of the BRST formalism in the framework of the augmented superfield formalism. Such a study is expected to shed some light on a few abstract mathematical structures behind the BRST formalism in a more intuitive and transparent manner. The usual superfield approach [18-25] to BRST formalism delves deep into providing the geometrical origin and interpretation for the conserved and nilpotent ($Q^2_{(a)b} = 0$) (anti-)BRST charges ($Q_{(a)b}$) which generate a set of local, covariant, continuous, nilpotent ($s^2_{(a)b} = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) (anti-)BRST symmetry transformations ($s_{(a)b}$) † *only* for the gauge field and the (anti-)ghost fields of the Lagrangian density of a given p-form ($p = 1, 2, 3....$) *interacting* Abelian gauge

*This set of theories is endowed with the first-class constraints in the language of the Dirac's prescription for the classification of constraints [1,2]. This observation is true for any arbitrary p-form gauge theories.

†We shall be following the notations and conventions adopted by Weinberg [26]. In its full glory, a nilpotent ($\delta_B^2 = 0$) BRST symmetry transformation δ_B is the product of an anticommuting spacetime independent parameter η and s_b as $\delta_B = \eta s_b$ with $s_b^2 = 0$. The parameter η anticommutes with all the fermionic fields (e.g. Dirac fields, (anti-)ghost fields, etc.) of a given interacting p-form gauge theory.

theory in the D-dimensions of spacetime. In this formalism [18-25], one constructs a $(p+1)$-form super curvature $\tilde{F} = \tilde{d}\tilde{A}$ by exploiting the super exterior derivative $\tilde{d}$ (with $\tilde{d}^2 = 0$) and the super connection p-from $\tilde{A}$ on a $(D+2)$-dimensional supermanifold parametrized by D-number of spacetime even coordinates $x^\mu (\mu = 0, 1, 2...D-1)$ and a couple of odd coordinates $\theta, \bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) of the Grassmann algebra. This super curvature $(p+1)$-form $\tilde{F}$ is subsequently equated, due to the so-called horizontality condition ‡, with the $(p+1)$-form ordinary curvature $F = dA$ constructed from the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and ordinary p-form connection A of a given p-form Abelian gauge theory §. In fact, the process of reduction of the (anti-)symmetric second-rank super tensor $\tilde{F}_{MN}$ (of $\tilde{F}$) to the ordinary antisymmetric second-rank tensor $F_{\mu\nu}$ (of F) produces the nilpotent (anti-)BRST symmetry transformations for the gauge- and the (anti-)ghost fields of the interacting p-form Abelian gauge theory. This restriction (i.e. $\tilde{F} = F$), however, does not shed any light on the nilpotent and anticommuting (anti-)BRST symmetry transformations for the *matter* fields of the interacting p-form Abelian gauge theory. Thus, it is clear that the derivation of the nilpotent symmetry transformations for *all* the fields, present in a given general p-form *interacting* (non-)Abelian gauge theory, is incomplete as far as the above superfield formulation [18-25] is concerned.

In our recent set of papers [28-31], the above usual superfield formulation [18-25,27] has been augmented by invoking the invariance of the conserved matter (super)currents on the (super)manifolds which leads to the derivation of the off-shell nilpotent symmetry transformations for the matter fields of the interacting 1-form gauge theory. We christen this extended version of the superfield formulation [28-31] as the augmented superfield formalism. The most interesting part of this prescription is the fact that there is a mutual consistency and complementarity between the requirements of (i) the horizontality condition, and (ii) the invariance of the matter (super)currents. The latter requirement is not imposed by hand from outside as is the case with the former (i.e. the horizontality condition). Rather, it turns out to be the inherent property of the interacting gauge theory itself. As the prototype examples of such a class of interacting gauge theories, we have chosen the 1-form interacting Abelian theories where (i) the Dirac fields couple to the Abelian gauge field A_μ in two $(1+1)$-dimensional (2D) spacetime, and (ii) the complex scalar fields and the Dirac fields couple to the Abelian $U(1)$ gauge field A_μ in four $(3+1)$-dimensions (4D) of spacetime. In the above field theoretical examples (i) and (ii), the equality (i.e.

‡This is referred to as the soul-flatness condition in [27] which amounts to setting equal to zero all the Grassmannian components of the $(p+1)$-rank (anti-)symmetric super curvature tensor defined on the $(D+2)$-dimensional supermanifold corresponding to a given p-form Abelian gauge theory.

§For the 1-form $(A = dx^\mu A_\mu \cdot T)$ non-Abelian gauge theory (see, e.g. [20]), the super 2-form curvature $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A} \wedge \tilde{A}$ (defined on the six $(4+2)$-dimensional supermanifold) is equated with the ordinary 2-form curvature $F = dA + A \wedge A$ (defined on the ordinary four $(3+1)$-dimensional Minkowskian submanifold of the above supermanifold) in the horizontality condition $\tilde{F} = F$ which leads to the exact derivation of the nilpotent and anticommuting (anti-)BRST symmetry transformations associated with the gauge (e.g. $s_b A_\mu = D_\mu C, s_{ab} A_\mu = D_\mu \bar{C}$) and (anti-)ghost fields (e.g. $s_b C = \frac{1}{2} C \times C, s_b \bar{C} = iB, s_{ab} C = i\bar{B}, s_{ab}\bar{C} = \frac{1}{2}\bar{C} \times \bar{C}$) of the four $(3+1)$-dimensional non-Abelian gauge theory (see, e.g., Sec. 3 below for the details).

$\tilde{J}_\mu(x,\theta,\bar{\theta}) = J_\mu(x)$) of the matter 2- and 4-vector supercurrent $\tilde{J}_\mu(x,\theta,\bar{\theta})$ (defined on the $(2+2)$- and $(4+2)$-dimensional supermanifolds, respectively) with the ordinary matter 2- and 4-vector conserved current $J_\mu(x)$ (defined on the $(1+1)$- and $(3+1)$-dimensional ordinary spacetime manifolds) leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the matter (i.e. the Dirac and the complex scalar) fields of the above chosen interacting gauge theories. These are found to be consistent with the nilpotent (anti-)BRST transformations for the gauge- and the (anti-)ghost fields derived from the horizontality condition. Furthermore, for the case of the field theoretic example in (i), the invariance of the axial-vector (super)currents (i.e. $\tilde{J}^{(5)}_\mu(x,\theta,\bar{\theta}) = J^{(5)}_\mu(x)$) on the (2+2)-dimensional (super)manifolds leads to the derivation of the nilpotent (anti-)co-BRST symmetry transformations for the Dirac fields which are checked to be consistent with the anticommuting and nilpotent (anti-)co-BRST transformations derived for the gauge- and the (anti-)ghost fields from the dual-horizontality condition (see, e.g., [28] for details).

The purpose of the present paper is to derive, in the framework of the augmented superfield formulation, the off-shell nilpotent version of the (anti-)BRST symmetry transformations for the Dirac (matter) fields of an interacting non-Abelian gauge theory where there is a coupling between the conserved matter vector current $J_\mu(x) = \bar{\psi}\gamma_\mu\psi$ and the group valued (i.e. $A_\mu = A^a_\mu T^a \equiv A_\mu \cdot T$) non-Abelian gauge field A_μ. We derive the off-shell (as well as on-shell) nilpotent symmetry transformations for the gauge- and the (anti-)ghost fields by exploiting the horizontality condition and the corresponding nilpotent symmetries for the matter fields are derived by invoking the invariance of the matter (super)currents defined on the (super)manifolds. We also derive separately (and independently) the BRST and anti-BRST symmetry transformations for the matter fields by invoking the invariance of the matter conserved (super)currents on the $(4+1)$-dimensional (anti-)chiral super sub-manifolds embedded in the most general $(4+2)$-dimensional supermanifold. The geometrical interpretation for the nilpotent (anti-)BRST charges $Q_{(a)b}$ as the translation generators on the six $(4+2)$-dimensional supermanifold is exactly same as the case of interacting Abelian gauge theories (see, e.g., [28,29]). The above interpretation is valid for the translations of all the superfields $(B_\mu, \Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ (cf.(3.1) and (4.1) below) corresponding to the gauge field, the (anti-)ghost fields and the matter fields (i.e. $A_\mu, C, \bar{C}, \psi, \bar{\psi}$) of the Lagrangian density (cf.(2.4) below) for the given 1-form interacting non-Abelian gauge theory in 4D. Our present study is essential primarily on three counts. First, it has been an outstanding problem to derive the nilpotent symmetries for the matter fields in the framework of superfield formulation. Second, through this work, we generalize our earlier works [28,29] on interacting Abelian gauge theories to the more general case of interacting non-Abelian gauge theory as the former is a limiting case of the latter. Third, to check the mutual consistency between the horizontality condition and the invariance of the matter (super)currents for the case of non-Abelian gauge theory which was found to be true for the case of interacting Abelian gauge theories in 2D and 4D (see, e.g., [28,29]).

The contents of our present paper are organized as follows. To set up the notations and

conventions, in section 2, we briefly sketch the off-shell (as well as the on-shell) nilpotent (anti-)BRST symmetry transformations for all the fields present in the Lagrangian density of an interacting non-Abelian gauge theory in four $(3+1)$-dimensions of spacetime. For the sake of the present paper to be self-contained, in section 3, we derive the off-shell (as well as the on-shell) nilpotent symmetry transformations for the gauge- and the (anti-)ghost fields in the framework of the usual superfield formulation by exploiting the horizontality condition (see, e.g.,[20,32,33] for details). The central result of the derivation of the nilpotent (anti-)BRST symmetry transformations for the Dirac fields is contained in section 4 where we exploit the invariance of the matter (super)currents on the most general (super)manifolds and the (anti-)chiral super sub-manifolds. Finally, we make some concluding remarks in section 5 and point out a few future directions which could be pursued later.

2 Nilpotent (Anti-)BRST Symmetries: Lagrangian Formalism

For the sake of simplicity, let us begin with the BRST invariant Lagrangian density $\mathcal{L}_b$ for the interacting 4D non-Abelian gauge theory ¶ in the Feynman gauge [26,27,34-36]

$$\mathcal{L}_b = -\frac{1}{4}\, F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi + B \cdot (\partial_\rho A^\rho) + \frac{1}{2}\, B \cdot B - i\, \partial_\mu \bar{C} \cdot D^\mu C, \qquad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + (A_\mu \times A_\nu)$ is the field strength tensor derived from the one-form connection $A = dx^\mu A_\mu \equiv dx^\mu A^a_\mu T^a = dx^\mu (A_\mu \cdot T)$ by the Maurer-Cartan equation $F = dA + A \wedge A$ with $F = \frac{1}{2} dx^\mu \wedge dx^\nu F^a_{\mu\nu} T^a$. Here T^a are the generators of the compact Lie algebra $[T^a, T^b] = i f^{abc} T^c$ where f^{abc} are the structure constants that can be chosen to be totally antisymmetric in a, b, c (see, e.g., [26] for details). The anticommuting ($(C^a)^2 = (\bar{C}^a)^2 = 0, C^a \bar{C}^b + \bar{C}^b C^a = 0, C^a \psi + \psi C^a = 0$ etc.) (anti-)ghost fields $(\bar{C}^a)C^a$ (which interact with the self-interacting non-Abelian gauge fields A_μ only in the loop diagrams) are required to be present in the theory to maintain the unitarity and the "quantum" gauge (i.e. BRST) invariance together at any arbitrary order of perturbative computations [37]. These fields (even though they do interact with the gauge fields A_μ) are not the physical matter fields. The physical matter fields of the theory are the Dirac (quark) fields $(\bar{\psi}, \psi)$ with the covariant derivative defined as ‖ : $D_\mu \psi = \partial_\mu \psi + i A_\mu \psi$ where the gauge field A_μ is group valued (i.e $A_\mu = A_\mu \cdot T$). The above Lagrangian density (2.1) respects the following off-shell nilpotent ($s_b^2 = 0$) BRST (s_b) symmetry transformations [26,27,34-36]

$$\begin{array}{c} s_b A_\mu = D_\mu C, \qquad s_b C = +\frac{1}{2} C \times C, \qquad s_b \bar{C} = iB, \qquad s_b B = 0, \\ s_b \psi = -i(C \cdot T)\psi, \qquad s_b \bar{\psi} = -i\bar{\psi}(C \cdot T), \qquad s_b F_{\mu\nu} = F_{\mu\nu} \times C. \end{array} \qquad (2.2)$$

¶We follow here the conventions and notations such that the flat 4D Minkowski metric $\eta_{\mu\nu}$ = diag $(+1, -1, -1, -1), D_\mu C = \partial_\mu C + A_\mu \times C, \alpha \cdot \beta = \alpha^a \beta^a, (\alpha \times \beta)^a = f^{abc} \alpha^b \beta^c$ where α and β are the non-null vectors in the group space. Here the Greek indices: $\mu, \nu, \rho ... = 0, 1, 2, 3$ correspond to the spacetime directions and Latin indices: $a, b, c... = 1, 2, 3...$ stand for the "colour" values in the group space.

‖Note that the coupling constant g in the actual definition of the covariant derivative $D_\mu \psi = \partial_\mu \psi + ig A_\mu \psi, D_\mu C = \partial_\mu C + g A_\mu \times C$ has been set equal to one (i.e. $g = 1$) for the sake of brevity.

It is worth pointing out, at this juncture, that (i) the kinetic energy term $-\frac{1}{4}(F^{\mu\nu} \cdot F_{\mu\nu})$ remains invariant under the BRST transformation s_b because of the presence of a totally antisymmetric f^{abc} in its variation (i.e. $s_b(-\frac{1}{4}F^{\mu\nu} \cdot F_{\mu\nu}) = -\frac{1}{2}f^{abc}F^{a\mu\nu}F^b_{\mu\nu}C^c = 0$). (ii) There exists an on-shell ($\partial_\mu D^\mu C = 0$) nilpotent ($\tilde{s}_b^2 = 0$) version of the above BRST symmetry transformations for this interacting non-Abelian gauge theory (see, e.g., [33])

$$\begin{array}{lll} \tilde{s}_b A_\mu = D_\mu C, & \tilde{s}_b C = +\frac{1}{2} C \times C, & \tilde{s}_b \bar{C} = -i(\partial_\mu A^\mu), \\ \tilde{s}_b \psi = -i(C \cdot T)\psi, & \tilde{s}_b \bar{\psi} = -i\bar{\psi}(C \cdot T), & \tilde{s}_b F_{\mu\nu} = F_{\mu\nu} \times C, \end{array} \tag{2.3}$$

under which, the following Lagrangian density for the interacting non-Abelian gauge theory

$$\mathcal{L}_{\tilde{b}} = -\tfrac{1}{4}\, F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi - \tfrac{1}{2}\, (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i\, \partial_\mu \bar{C} \cdot D^\mu C, \tag{2.4}$$

transforms to a total derivative. It will be noted that (2.4) and (2.3) have been derived from (2.1) and (2.2), respectively, by exploiting the equation of motion $B + (\partial_\mu A^\mu) = 0$ emerging from (2.1). (iii) Besides the symmetry transformations (2.2) and (2.3), there exists a nilpotent ($s_{ab}^2 = 0$) anti-BRST symmetry transformation (s_{ab}) that is also present in the theory, in a subtle way. To realize this anti-BRST symmetry, one has to introduce another auxiliary field $\bar{B}$ (satisfying $B + \bar{B} = i\; C \times \bar{C}$) to recast the Lagrangian density (2.1) into the following equivalent forms (see, e.g., [38,34-36])

$$\begin{array}{lll} \mathcal{L}_{\bar{B}} & = & -\frac{1}{4}\, F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + B \cdot (\partial_\mu A^\mu) \\ & + & \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - i\partial_\mu \bar{C} \cdot D^\mu C, \end{array} \tag{2.5a}$$

$$\begin{array}{lll} \mathcal{L}_{\bar{B}} & = & -\frac{1}{4}\, F^{\mu\nu} \cdot F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \bar{B} \cdot (\partial_\mu A^\mu) \\ & + & \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - iD_\mu \bar{C} \cdot \partial^\mu C. \end{array} \tag{2.5b}$$

The Lagrangian density (2.5b) transforms to a total derivative under the following off-shell nilpotent ($s_{ab}^2 = 0$) anti-BRST (s_{ab}) symmetry transformations (see, e.g., [34,26] for details)

$$\begin{array}{lllll} s_{ab} A_\mu & = & D_\mu \bar{C}, & s_{ab} \bar{C} = +\frac{1}{2} \bar{C} \times \bar{C}, & s_{ab} C = i\bar{B}, \quad s_{ab} B = B \times \bar{C}, \\ s_{ab} F_{\mu\nu} & = & F_{\mu\nu} \times \bar{C}, & s_{ab} \psi = -i(\bar{C} \cdot T)\psi, & s_{ab} \bar{\psi} = -i\bar{\psi}(\bar{C} \cdot T), \quad s_{ab} \bar{B} = 0. \end{array} \tag{2.6}$$

It can be explicitly verified, using (2.2) and (2.6), that $s_b s_{ab} + s_{ab} s_b = 0$ for any arbitrary field of (2.1). For such a verification, the transformation (2.2) should be extended to include $s_b \bar{B} = \bar{B} \times C$. At this stage, it is worthwhile to point out that, unlike the BRST symmetries (cf. (2.2) and (2.3)), there is no on-shell nilpotent version of the anti-BRST symmetry transformations in (2.6). All the above continuous symmetry transformations can be concisely expressed, in terms of the Noether conserved and the off-shell (as well as on-shell) nilpotent ($Q_r^2 = 0, \tilde{Q}_r^2 = 0$) charges $Q_r(\tilde{Q}_r)$ as

$$s_r \Sigma(x) = -i\; [\Sigma(x), Q_r]_\pm, \qquad r = b, ab, \qquad \tilde{s}_r \Sigma(x) = -i\; [\Sigma(x), \tilde{Q}_r]_\pm, \qquad r = b, \tag{2.7}$$

where the $\pm$ signs on the brackets stand for the brackets to be an (anti-)commutator for the generic field $\Sigma = A_\mu, C, \bar{C}, \psi, \bar{\psi}, B, \bar{B}$ being (fermionic)bosonic in nature. The transformations $\tilde{s}_r$, in the above, correspond to the on-shell nilpotent BRST transformations (2.3).

3 Nilpotent Symmetries for Gauge- and (Anti-)ghost Fields: Superfield Formalism

We recapitulate here the bare essentials of some of the key points connected with the idea of the horizontality condition and its application. To this end in mind, first of all, let us generalize the generic local field $\Sigma(x) = (A_\mu, C, \bar{C})(x)$ corresponding to the gauge- and the (anti-)ghost fields of the 4D Lagrangian density (2.1) to a supervector superfield $(\tilde{A}_M \cdot T)(x, \theta, \bar{\theta}) = (B_\mu \cdot T, \Phi \cdot T, \bar{\Phi} \cdot T)(x, \theta, \bar{\theta})$ defined on the six $(4+2)$-dimensional supermanifold with the following super expansion (see, e.g., [20,32,33] for details)

$$
\begin{array}{lcl}
(B_\mu \cdot T)(x,\theta,\bar{\theta}) & = & (A_\mu \cdot T)(x) + \bar{\theta}\,(R_\mu \cdot T)(x) + \theta\,(\bar{R}_\mu \cdot T)(x) + i\,\theta\,\bar{\theta}\,(S_\mu \cdot T)(x),\\
(\Phi \cdot T)(x,\theta,\bar{\theta}) & = & (C \cdot T)(x) + i\,\bar{\theta}\,(\mathcal{B} \cdot T)(x) + i\,\theta\,(\bar{B} \cdot T)(x) + i\,\theta\,\bar{\theta}\,(s \cdot T)(x),\\
(\bar{\Phi} \cdot T)(x,\theta,\bar{\theta}) & = & (\bar{C} \cdot T)(x) + i\,\bar{\theta}\,(B \cdot T)(x) + i\,\theta\,(\bar{\mathcal{B}} \cdot T)(x) + i\,\theta\,\bar{\theta}\,(\bar{s} \cdot T)(x).
\end{array}
\tag{3.1}
$$

The noteworthy points, at this stage, are (i) all the superfields $(\tilde{A}_M \cdot T)(x, \theta, \bar{\theta}) = (B_\mu \cdot T, \Phi \cdot T, \bar{\Phi} \cdot T)(x, \theta, \bar{\theta})$ as well as local fields (e.g. $A_\mu = A_\mu \cdot T, C = C \cdot T$ etc.) are group valued; (ii) the degrees of freedom of the group valued ** fermionic (odd) fields $R_\mu, \bar{R}_\mu, C, \bar{C}, s, \bar{s}$ match with that of the bosonic (even) fields $A_\mu, S_\mu, B, \bar{B}, \mathcal{B}, \bar{\mathcal{B}}$ so that the theory can be consistent with the basic requirements of supersymmetry; and (iii) the horizontality restriction $\tilde{F} = \tilde{D}\tilde{A} = DA = F$ (where $\tilde{D}\tilde{A} = \tilde{d}\tilde{A} + \tilde{A} \wedge \tilde{A}, DA = dA + A \wedge A$) leads to the following relationships (see, e.g., [20,32,33] for details)

$$
\begin{array}{lcl}
R_\mu(x) & = & D_\mu C(x), \quad \bar{R}_\mu(x) = D_\mu \bar{C}(x), \quad B(x) + \bar{B}(x) = i\,(C \times \bar{C})(x),\\
\mathcal{B}(x) & = & -\frac{i}{2}\,(C \times C)(x), \quad \bar{\mathcal{B}}(x) = -\frac{i}{2}\,(\bar{C} \times \bar{C})(x), \quad \bar{s}(x) = -(B \times \bar{C})(x),\\
S_\mu(x) & = & D_\mu B(x) - (D_\mu C \times \bar{C})(x), \qquad s(x) = (\bar{B} \times C)(x),
\end{array}
\tag{3.2}
$$

where $S_\mu(x)$ can be equivalently written as: $S_\mu(x) = -D_\mu \bar{B}(x) - (D_\mu \bar{C} \times C)(x)$ and the individual terms in $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A} \wedge \tilde{A}$ can be explicitly expressed as

$$
\begin{array}{lcl}
\tilde{d}\tilde{A} & = & (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) + (dx^\mu \wedge d\theta)(\partial_\mu \bar{\Phi} - \partial_\theta B_\mu) - (d\theta \wedge d\theta)(\partial_\theta \bar{\Phi})\\
& + & (dx^\mu \wedge d\bar{\theta})(\partial_\mu \Phi - \partial_{\bar{\theta}} B_\mu) - (d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}} \Phi) - (d\theta \wedge d\bar{\theta})(\partial_\theta \Phi + \partial_{\bar{\theta}} \bar{\Phi}),\\
\tilde{A} \wedge \tilde{A} & = & (dx^\mu \wedge dx^\nu)(B_\mu B_\nu) + (dx^\mu \wedge d\theta)([B_\mu, \bar{\Phi}]) - (d\theta \wedge d\theta)(\bar{\Phi}\bar{\Phi}),\\
& + & (dx^\mu \wedge d\bar{\theta})([B_\mu, \Phi]) - (d\bar{\theta} \wedge d\bar{\theta})(\Phi\Phi) - (d\theta \wedge d\bar{\theta})(\{\Phi, \bar{\Phi}\}).
\end{array}
\tag{3.3}
$$

It is interesting to point out that the condition $(B(x) + \bar{B}(x) = i(C \times \bar{C}))$ (see, e.g., [38]), required for the definition of the anti-BRST symmetry transformations s_{ab} (cf. section 2), emerges here automatically in the superfield formulation. In the above computation, the basic super derivative $\tilde{d}$ and 1-form super connection $\tilde{A}$ are defined on the six $(4+2)$-dimensional supermanifold, in terms of the superspace differentials $dx^\mu, d\theta$ and $d\bar{\theta}$, as

$$
\begin{array}{lcl}
\tilde{d} & = & dZ^M\,\partial_M \equiv dx^\mu\,\partial_\mu + d\theta\,\partial_\theta + d\bar{\theta}\,\partial_{\bar{\theta}},\\
\tilde{A} & = & dZ^M\,(\tilde{A}_M \cdot T) \equiv dx^\mu\,(B_\mu \cdot T) + d\theta\,(\bar{\Phi} \cdot T) + d\bar{\theta}\,(\Phi \cdot T),
\end{array}
\tag{3.4}
$$

**Hereafter, we shall be shuffling between the explicit group valued notations (e.g. $A_\mu = A_\mu \cdot T, R_\mu = R_\mu \cdot T, s = s \cdot T$ etc.) and their shorter versions (e.g. A_μ, R_μ, s etc.) in the remaining part of the paper.

where $Z^M = (x^\mu, \theta, \bar\theta)$ and $\partial_M = (\partial/\partial Z^M)$ are the generic superspace coordinates and the corresponding superspace derivatives on the six (4 + 2)-dimensional supermanifold. Ultimately, the insertions of the values from (3.2) and the use of the (anti-)BRST transformations of equations (2.6) and (2.2) lead to the following form for the expansion (3.1):

$$\begin{array}{lll} B_\mu(x,\theta,\bar\theta) & = & A_\mu(x) + \theta\,(s_{ab} A_\mu(x)) + \bar\theta\,(s_b A_\mu(x)) + \theta\,\bar\theta\,(s_b s_{ab} A_\mu(x)), \\ \Phi(x,\theta,\bar\theta) & = & C(x) + \theta\,(s_{ab} C(x)) + \bar\theta\,(s_b C(x)) + \theta\,\bar\theta\,(s_b s_{ab} C(x)), \\ \bar\Phi(x,\theta,\bar\theta) & = & \bar C(x) + \theta\,(s_{ab} \bar C(x)) + \bar\theta\,(s_b \bar C(x)) + \theta\,\bar\theta\,(s_b s_{ab} \bar C(x)). \end{array} \tag{3.5}$$

In the above equation, the shorter (group valued) notations (i.e. $B_\mu = B_\mu \cdot T, \Phi = \Phi \cdot T, \bar\Phi = \bar\Phi \cdot T$ etc.) have been used for expansions along the Grassmannian directions. It is clear from (3.5) that the (anti-)BRST charges $Q_{(a)b}$ do correspond to the translation generators $(\text{Lim}_{\bar\theta\to 0}(\partial/\partial\theta))$ $\text{Lim}_{\theta\to 0}(\partial/\partial\bar\theta)$ along the Grassmannian directions of the supermanifold.

To dwell a bit on the derivation of the on-shell nilpotent BRST symmetry transformations (2.3) for the gauge- and the (anti-)ghost fields, we recapitulate the bare essentials of our earlier works [32,33]. To this end in mind, we begin with the chiral (i.e. $\theta \to 0$) limit of (i) the expansions in (3.1), and (ii) the definitions in (3.4) as follows [32,33]

$$\begin{array}{lll} B^{(c)}_\mu(x,\bar\theta) & = & A_\mu(x) + \bar\theta\, R_\mu(x), \qquad \Phi^{(c)}(x,\bar\theta) = C(x) + i\,\bar\theta\,\mathcal{B}(x), \\ \bar\Phi^{(c)}(x,\bar\theta) & = & \bar C(x) + i\,\bar\theta\, B(x), \qquad \tilde d|_{(c)} = dx^\mu\,\partial_\mu + d\bar\theta\,\partial_{\bar\theta}, \\ \tilde A|_{(c)}(x,\bar\theta) & = & dx^\mu\, B^{(c)}_\mu(x,\bar\theta) + d\bar\theta\,\Phi^{(c)}(x,\bar\theta), \end{array} \tag{3.6}$$

where, for the sake of brevity, we have taken the help of the shorter version of our notations where $B^{(c)}_\mu = (B^{(c)}_\mu \cdot T), \Phi^{(c)} = (\Phi^{(c)} \cdot T), \bar\Phi^{(c)} = (\bar\Phi^{(c)} \cdot T)$, etc. The application of the horizontality condition in terms of the chiral super 1-form connection and super derivative (i. e. $\tilde d|_{(c)} \tilde A|_{(c)} + \tilde A|_{(c)} \wedge \tilde A_{(c)} = dA + A \wedge A$) yields the following relationships between the secondary fields and the basic fields (see [32,33] for all the details)

$$R_\mu(x) = D_\mu C(x), \qquad\qquad \mathcal{B}(x) = -\tfrac{i}{2}\,(C \times C)(x), \tag{3.7}$$

where the individual terms on the left hand side $(\tilde d|_{(c)} \tilde A|_{(c)} + \tilde A|_{(c)} \wedge \tilde A_{(c)})$ of the above horizontality condition possess the following explicit forms

$$\begin{array}{lll} \tilde d|_{(c)} \tilde A|_{(c)} & = & (dx^\mu \wedge dx^\nu)(\partial_\mu B^{(c)}_\nu) + (dx^\mu \wedge d\bar\theta)(\partial_\mu \Phi^{(c)} - \partial_{\bar\theta} B^{(c)}_\mu), \\ & - & (d\bar\theta \wedge d\bar\theta)(\partial_{\bar\theta}\Phi^{(c)}), \\ \tilde A|_{(c)} \wedge \tilde A|_{(c)} & = & (dx^\mu \wedge dx^\nu)(B^{(c)}_\mu B^{(c)}_\nu) + (dx^\mu \wedge d\bar\theta)([B^{(c)}_\mu, \Phi^{(c)}]), \\ & - & (d\bar\theta \wedge d\bar\theta)(\Phi^{(c)}\Phi^{(c)}). \end{array} \tag{3.8}$$

It is evident that the above horizontality restriction does not shed any light on the auxiliary field $B = (B \cdot T)$. However, the equation of motion $B + (\partial_\mu A^\mu) = 0$, emerging from the Lagrangian density (2.1), provides the relationship between the auxiliary field B and the basic field A_μ. The insertion of all these values in the chiral expansion (3.6) (*vis-à-vis* the symmetry transformations in (2.3)) leads to

$$\begin{array}{lll} B^{(c)}_\mu(x,\bar\theta) & = & A_\mu(x) + \bar\theta\,(\tilde s_b A_\mu(x)), \qquad \Phi^{(c)}(x,\bar\theta) = C(x) + \bar\theta\,(\tilde s_b C(x)), \\ \bar\Phi^{(c)}(x,\bar\theta) & = & \bar C(x) + \bar\theta\,(\tilde s_b \bar C(x)). \end{array} \tag{3.9}$$

The above expansion clearly establishes that the on-shell ($\partial_\mu D^\mu C = 0$) nilpotent ($\tilde{Q}_b^2 = 0$) BRST charge $\tilde{Q}_b$ corresponds to the translation ($\partial/\partial\bar{\theta}$) along the $\bar{\theta}$-direction of the $(4+1)$-dimensional chiral super sub-manifold parametrized by the spacetime variable x^μ and a Grassmannian variable $\bar{\theta}$. We would like to lay emphasis on the fact that the anti-chiral (i.e.$\bar{\theta} \to 0$) limit of (i) the expansion in (3.1), and (ii) the definitions in (3.4) does not lead to any worthwhile symmetries. Thus, in some sense, the superfield formalism does shed some light on the absence of the on-shell nilpotent anti-BRST symmetry transformations for the non-Abelian gauge theory in any dimensions of spacetime. This observation should be contrasted with the Abelian gauge theory where the on-shell nilpotent (anti-)BRST symmetries do exist for the Lagrangian density of the Abelian gauge theory (see, e.g., [32] for details). We shall discuss more about the geometrical interpretations of some of the key properties associated with $Q_{(a)b}$ in section 5 of our present paper.

4 Nilpotent Symmetries for Dirac Fields: Augmented Superfield Approach

To obtain the nilpotent (anti-)BRST transformations for the Dirac fields (cf. (2.6), (2.3) and (2.2)) in the framework of the augmented superfield formalism, we begin with the following super expansion for the superfields $\Psi(x,\theta,\bar{\theta}), \bar{\Psi}(x,\theta,\bar{\theta})$ corresponding to the ordinary Dirac fields $\psi(x), \bar{\psi}(x)$ of Lagrangian density (2.1):

$$\begin{array}{lll} \Psi(x,\theta,\bar{\theta}) & = & \psi(x) + i\,\theta\;(\bar{b}_1 \cdot T)(x) + i\,\bar{\theta}\,(b_2 \cdot T)(x) + i\,\theta\,\bar{\theta}\,(f \cdot T)(x), \\ \bar{\Psi}(x,\theta,\bar{\theta}) & = & \bar{\psi}(x) + i\,\theta\,(\bar{b}_2 \cdot T)(x) + i\,\bar{\theta}\,(b_1 \cdot T)(x) + i\,\theta\,\bar{\theta}\,(\bar{f} \cdot T)(x). \end{array} \tag{4.1}$$

A few comments are in order now. First, it is clear from the above that the basic tenets of supersymmetry are satisfied here because the number of degrees of freedom of the bosonic (i.e. $b_1, \bar{b}_1, b_2, \bar{b}_2$) and the fermionic (i.e. $\psi, \bar{\psi}, f, \bar{f}$) fields do match here. Second, in general, these fields are group valued (i.e. $b_1 \equiv b_1 \cdot T, b_2 \equiv b_2 \cdot T, f \equiv f \cdot T$, etc.). Third, in the limit $\theta, \bar{\theta} \to 0$, we obtain the ordinary fermionic fields (i.e. Dirac fields) $\psi(x)$ and $\bar{\psi}(x)$. Fourth, unlike the horizontality condition (i.e. $\tilde{F} = F$) of the previous section (which involves the use of (super)exterior derivatives $(\tilde{d})d$ and the 1-form (super)connections $(\tilde{A})A$), we exploit here the invariance of the (super)currents on the (super)manifolds. To this end in mind, we construct here the supercurrent $\tilde{J}_\mu(x,\theta,\bar{\theta})$ with the fermionic superfields of (4.1) as

$$\tilde{J}_\mu(x,\theta,\bar{\theta}) = \bar{\Psi}(x,\theta,\bar{\theta})\;\gamma_\mu\;\Psi(x,\theta,\bar{\theta}) = J_\mu(x) + \theta\;\bar{K}_\mu(x) + \bar{\theta}\;K_\mu(x) + i\;\theta\;\bar{\theta}\;L_\mu(x), \tag{4.2}$$

where the bosonic components $J_\mu(x), L_\mu(x)$ are along the $\hat{\mathbf{1}}$ and $\theta\bar{\theta}$-directions of the supermanifold and the fermionic components $\bar{K}_\mu(x), K_\mu(x)$ are along the Grassmannian directions θ and $\bar{\theta}$ of the supermanifold. These components (with $J_\mu(x) = \bar{\psi}\gamma_\mu\psi$) can be expressed in terms of the components of the basic expansion (4.1) as

$$\begin{array}{l} \bar{K}_\mu(x) = i\;[(\bar{b}_2 \cdot T)\gamma_\mu\psi - \bar{\psi}\gamma_\mu(\bar{b}_1 \cdot T)], \qquad K_\mu(x) = i\;[(b_1 \cdot T)\gamma_\mu\psi - \bar{\psi}\gamma_\mu(b_2 \cdot T)], \\ L_\mu(x) = (\bar{f} \cdot T)\gamma_\mu\psi + \bar{\psi}\gamma_\mu(f \cdot T) + i\;[(\bar{b}_2 \cdot T)\gamma_\mu(b_2 \cdot T) - (b_1 \cdot T)\gamma_\mu(\bar{b}_1 \cdot T)]. \end{array} \tag{4.3}$$

To be consistent with the following inter-relationships

$$s_b \leftrightarrow \text{Lim}_{\theta\to 0}(\partial/\partial\bar\theta) \leftrightarrow Q_b, \qquad\qquad s_{ab} \leftrightarrow \text{Lim}_{\bar\theta\to 0}(\partial/\partial\theta) \leftrightarrow Q_{ab}. \tag{4.4}$$

that were established earlier (cf. section 3) in the context of the derivation of the off-shell nilpotent transformations for the (bosonic) gauge- and the (fermionic) (anti-)ghost fields, it is interesting to re-express (4.2) as given below

$$\tilde J_\mu(x,\theta,\bar\theta) = J_\mu(x) + \theta\,(s_{ab}J_\mu(x)) + \bar\theta\,(s_b J_\mu(x)) + \;\theta\,\bar\theta\,(s_b s_{ab} J_\mu(x)). \tag{4.5}$$

It can be explicitly checked, however, that $s_{(a)b}J_\mu(x) = 0$ if we exploit the exact form of nilpotent (anti-)BRST transformations of (2.6) (2.3) and (2.2) for the matter fields ψ and $\bar\psi$. Thus, the *natural* restriction that emerges on the six $(4+2)$-dimensional (super)manifolds is: $\tilde J_\mu(x,\theta,\bar\theta) = J_\mu(x)$. In the physical language, this restriction implies that there are no superspace (i.e. Grassmannian) contribution to the conserved supercurrent and it is identically equal to the usual conserved current $J_\mu(x) = \bar\psi\gamma_\mu\psi$ on the 4D ordinary manifold. This natural requirement, in turn, implies the following conditions:

$$\begin{array}{ll} (b_1\cdot T)\,\gamma_\mu\,\psi = \bar\psi\,\gamma_\mu\,(b_2\cdot T), & (\bar b_2\cdot T)\,\gamma_\mu\,\psi = \bar\psi\,\gamma_\mu\,(\bar b_1\cdot T), \\ (\bar f\cdot T)\,\gamma_\mu\,\psi + \bar\psi\,\gamma_\mu\,(f\cdot T) = i\,[\,(b_1\cdot T)\,\gamma_\mu\,(\bar b_1\cdot T) - (\bar b_2\cdot T)\,\gamma_\mu\,(b_2\cdot T)\,]. & \end{array} \tag{4.6}$$

The above conditions are to be satisfied by the components of the basic expansion (4.1). In other words, we have to find the solution to the above restrictions so that the components of (4.1) could be expressed in terms of the basic fields of the Lagrangian density (2.1) or (2.4). Such a solution of our interest is given below

$$\begin{array}{ll} b_1 \equiv b_1\cdot T = -\bar\psi(C\cdot T), & b_2 \equiv b_2\cdot T = -(C\cdot T)\,\psi, \\ \bar b_1 \equiv \bar b_1\cdot T = -(\bar C\cdot T)\psi, & \bar b_2 \equiv \bar b_2\cdot T = -\bar\psi(\bar C\cdot T), \\ f \equiv f\cdot T = -\,i\,[B\cdot T + \frac{1}{2}\,(C\times\bar C)\cdot T]\,\psi(x), & \\ \bar f \equiv \bar f\cdot T = \;i\,\bar\psi(x)\,[B\cdot T + \frac{1}{2}\,(C\times\bar C)\cdot T]. & \end{array} \tag{4.7}$$

It can be seen from $b_1\gamma_\mu\psi = \bar\psi\gamma_\mu b_2$ that this equality could be satisfied if and only if b_1 and b_2 were proportional to $\bar\psi$ and ψ, respectively. However, the bosonic and group-valued nature of b_1 and b_2 enforces that the group-valued fermionic ghost fields should be brought in to make this equality work. This is why we take $b_1 = -\bar\psi\,(C\cdot T)$ and $b_2 = -(C\cdot T)\,\psi$. The rest of the choices in (4.7) are made along similar lines of argument. Insertion of the above values of the bosonic components $b_1, \bar b_1, b_2, \bar b_2$ and the fermionic components f and $\bar f$ in the basic super expansion (4.1) leads to the following

$$\begin{array}{lll} \Psi(x,\theta,\bar\theta) & = & \psi(x) + \;\theta\,(s_{ab}\psi(x)) + \;\bar\theta\,(s_b\psi(x)) + \;\theta\,\bar\theta\,(s_b s_{ab}\psi(x)), \\ \bar\Psi(x,\theta,\bar\theta) & = & \bar\psi(x) + \;\theta\,(s_{ab}\bar\psi(x)) + \;\bar\theta\,(s_b\bar\psi(x)) + \;\theta\,\bar\theta\,(s_b s_{ab}\bar\psi(x)). \end{array} \tag{4.8}$$

This establishes the sanctity of the mappings in (4.4). In fact, these mappings are valid for all the fields of the interacting 1-form non-Abelian gauge theory in 4D. It is also clear that there is a mutual consistency and complementarity between the horizontality condition and

the invariance of the conserved matter (super)currents of the theory on the most general $(4+2)$-dimensional (super)manifolds.

It is worthwhile to mention, before we wrap up this section, that the off-shell nilpotent (anti-)BRST transformations for the matter fields can be derived separately (and independently) in a very simple manner. For this purpose, we first take the chiral (i.e. $\theta \to 0$) limit of the expansion in (4.1) to obtain an expansion on a five $(4+1)$-dimensional chiral super sub-manifold, embedded in the most general $(4+2)$-dimensional supermanifold, as

$$\Psi^{(c)}(x,\bar{\theta}) \;=\; \psi(x) + i\,\bar{\theta}\;(b_2 \cdot T)(x), \qquad \bar{\Psi}^{(c)}(x,\bar{\theta}) = \bar{\psi}(x) + i\,\bar{\theta}\,(b_1 \cdot T)(x). \tag{4.9}$$

We can now construct the chiral super current, using the above expansion, as follows

$$\tilde{J}^{(c)}_\mu(x,\bar{\theta}) = \bar{\Psi}^{(c)}(x,\bar{\theta})\;\gamma_\mu\;\Psi^{(c)}(x,\bar{\theta}) = J_\mu(x) + i\,\bar{\theta}\,[(b_1 \cdot T)\gamma_\mu\psi - \bar{\psi}\gamma_\mu(b_2 \cdot T)]. \tag{4.10}$$

Exploiting the natural restriction, discussed earlier in great detail, that the chiral supercurrent $\tilde{J}^{(c)}_\mu(x,\bar{\theta})$ defined on the chiral super sub-manifold should be equal (i.e. $\tilde{J}^{(c)}_\mu(x,\bar{\theta}) = J_\mu(x)$) to the ordinary current $J_\mu(x)$, we obtain the following expressions for b_1 and b_2 in terms of the basic fields of the theory:

$$b_1 \equiv b_1 \cdot T = -\bar{\psi}(C \cdot T), \qquad b_2 \equiv b_2 \cdot T = -(C \cdot T)\;\psi. \tag{4.11}$$

The substitution of these values in the chiral expansion (4.9) leads to the derivation of the BRST symmetry transformations for the matter fields as

$$\Psi^{(c)}(x,\bar{\theta}) \;=\; \psi(x) + \;\bar{\theta}\;(s_b\psi(x)), \qquad \bar{\Psi}^{(c)}(x,\bar{\theta}) = \bar{\psi}(x) + \;\bar{\theta}\;(s_b\bar{\psi}(x)). \tag{4.12}$$

Exploiting (2.7), it is clear that the generator Q_b of the off-shell nilpotent BRST symmetry transformations is the generator of translation $(\partial/\partial\bar{\theta})$ along the Grassmannian direction $\bar{\theta}$ of the $(4+1)$-dimensional super sub-manifold. In an analogous manner, one can derive the anti-BRST symmetries for the matter fields by taking the anti-chiral (i.e. $\bar{\theta} \to 0$) limit of the super expansion in (4.1) as given below

$$\Psi^{(ac)}(x,\theta) \;=\; \psi(x) + i\,\theta\;(\bar{b}_1 \cdot T)(x), \qquad \bar{\Psi}^{(ac)}(x,\bar{\theta}) = \bar{\psi}(x) + i\,\theta\,(\bar{b}_2 \cdot T)(x). \tag{4.13}$$

In terms of these expansion, the anti-chiral super current $\tilde{J}^{(ac)}_\mu(x,\theta)$, defined on the $(4+1)$-dimensional anti-chiral super sub-manifold, can be written as follows

$$\tilde{J}^{(ac)}_\mu(x,\theta) = \bar{\Psi}^{(ac)}(x,\theta)\;\gamma_\mu\;\Psi^{(ac)}(x,\theta) = J_\mu(x) + i\theta\,[(\bar{b}_2 \cdot T)\gamma_\mu\psi - \bar{\psi}\gamma_\mu(\bar{b}_1 \cdot T)]. \tag{4.14}$$

The natural restriction $\tilde{J}^{(ac)}_\mu(x,\theta) = J_\mu(x)$ on the anti-chiral $(4+1)$-dimensional super sub-manifold implies the following expressions for $\bar{b}_1$ and $\bar{b}_2$ in terms of the basic fields of the Lagrangian density (2.1) of the theory:

$$\bar{b}_1 \equiv \bar{b}_1 \cdot T = -(\bar{C} \cdot T)\psi, \qquad \bar{b}_2 \equiv \bar{b}_2 \cdot T = -\bar{\psi}(\bar{C} \cdot T). \tag{4.15}$$

The insertions of these values in the super expansion (4.13) leads to the derivation of the anti-BRST symmetry transformations for the matter fields as illustrated below

$$\Psi^{(ac)}(x,\bar{\theta}) \;=\; \psi(x) + \;\theta\;\, (s_{ab}\psi(x)), \qquad \bar{\Psi}^{(ac)}(x,\bar{\theta}) = \bar{\psi}(x) + \;\theta\;(s_{ab}\bar{\psi}(x)). \qquad (4.16)$$

This establishes the geometrical interpretation for the nilpotent anti-BRST charge Q_{ab} as the translation generator $(\partial/\partial\theta)$ along the Grassmannian θ-direction of the anti-chiral $(4+1)$-dimensional super sub-manifold. Physically, the process of translation of the super fields $\Psi(x,\theta)$ and $\bar{\Psi}(x,\theta)$ along the θ-direction of the five $(4+1)$-dimensional anti-chiral super sub-manifold generates the off-shell nilpotent anti-BRST symmetry transformations for the ordinary Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ of the Lagrangian density (2.1) of the theory.

5 Conclusions

We have exploited in our present investigation (i) the horizontality condition, and (ii) the invariance of the conserved matter (super)currents on a set of (super)manifolds. These supermanifolds are of (1) a general nature with $(4+2)$-dimensional superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$, and (2) a special variety with $(4+1)$-dimensional (anti-)chiral superspace variables $Z^M = (x^\mu, \theta)$ and/or $Z^M = (x^\mu, \bar{\theta})$. The above physically motivated restrictions (i) and (ii), which are the salient features of the augmented superfield formulation, lead to primarily *four* key consequences. First, they provide the geometrical interpretation for the conserved and off-shell nilpotent (anti-)BRST charges $Q_{(a)b}$ as the translation generators $((\mathrm{Lim}_{\bar{\theta}\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta}))$ along the Grassmannian directions $(\theta)\bar{\theta}$ of the most *general* $(4+2)$-dimensional supermanifold. Second, they produce *together* the off-shell nilpotent (anti-)BRST symmetry transformations for the gauge-, the (anti-)ghost- and the matter fields of an interacting gauge theory. Third, the anticommutativity $(s_b s_{ab} + s_{ab} s_b = 0)$ of the (anti-)BRST symmetries (as well as corresponding charges) is expressed in the language of the translation generators because $(\partial/\partial\theta)(\partial/\partial\bar{\theta}) + (\partial/\partial\bar{\theta})(\partial/\partial\theta) = 0$. Finally, they furnish the geometrical interpretation for the nilpotency property (i.e. $s_{(a)b}^2 = 0$) of the (anti-)BRST transformations (and their corresponding nilpotent generators) in terms of a couple of successive translations (i.e. $(\partial/\partial\theta)^2 = 0, (\partial/\partial\bar{\theta})^2 = 0$) along either of the Grassmannian directions $(\theta)\bar{\theta}$ of the $(4+2)$-dimensional supermanifold.

As a side remark, it is interesting to point out that the *on-shell* nilpotent version of the symmetry transformations for the gauge- and the (anti-)ghost fields have been derived by exploiting the horizontality condition on the $(4+1)$-dimensional (anti-)chiral super sub-manifolds, embedded in the general $(4+2)$-dimensional supermanifold. For instance, on the $(4+1)$-dimensional chiral super sub-manifold, the on-shell $(\partial_\mu D^\mu C = 0)$ nilpotent $(\tilde{Q}_b^2 = 0)$ BRST charge $\tilde{Q}_b$ for the non-Abelian gauge theory corresponds to the translation generator $(\partial/\partial\bar{\theta})$ along the $\bar{\theta}$-direction of the super sub-manifold (cf. section 3). There exists no such on-shell nilpotent anti-BRST symmetry in the present non-Abelian gauge theory (as emphasized in section 2). The superfield formulation sheds some light on the reasons behind the non-existence of the on-shell nilpotent anti-BRST symmetries for the

non-Abelian gauge theory (see, e.g., [32] for details). In contrast, the requirement of the invariance of the (super)currents on (i) the $(4+2)$-dimensional supermanifold, and (ii) the $(4+1)$-dimensional (anti-)chiral super sub-manifolds, leads to the derivation of the *off-shell* nilpotent and anticommuting (anti-)BRST symmetry transformations for the matter fields of the interacting non-Abelian gauge theory. In the first case, we obtain these (anti-)BRST symmetries together and, in the latter case, we obtain these symmetries separately and independently. The geometrical interpretation for the (anti-)BRST charges $Q_{(a)b}$ as the translation generators along $(\theta)\bar{\theta}$-directions of the supermanifold remains intact for the cases of the general supermanifold as well as the (anti-)chiral supermanifolds.

One of the most interesting features of the above restrictions on the supermanifolds is the mutual consistency and complementarity between them. These restrictions are at the heart of a complete geometrical description of all the nilpotent symmetry transformations for all the fields present in an interacting 1-form non-Abelian gauge theory. It is worthwhile to mention that, physically, the invariance ($\tilde{J}_\mu(x,\theta,\bar{\theta}) = J_\mu(x)$) of the gauge invariant and conserved matter (super)currents on the (super)manifolds implies the conservation of charge which, ultimately, does not get any contribution from the superspace (Grassmannian) variables. This restriction turns out to be the *natural* one on the supermanifold. On the other hand, physically, the horizontality condition owes its origin to the gauge invariance of the electric and magnetic fields for the Abelian gauge theory [28,29] and the gauge invariance of the kinetic energy term ($-\frac{1}{4}F^{\mu\nu} \cdot F_{\mu\nu}$) for the case of the non-Abelian gauge theory. Thus, the horizontality condition ($\tilde{F} = F$) requires that the Grassmannian (i.e. θ and $\bar{\theta}$) contribution to the 2-form curvature field should be *zero* because the second-rank tensor associated with it corresponds to the gauge invariant physical electric- and magnetic fields. Mathematically, the equality $\tilde{F} = F$ owes its origin to the cohomological (super) exterior derivatives $(\tilde{d})d$ which play a very decisive role in the definition of the (super) 2-forms $(\tilde{F})F$. In this connection, it is pertinent to point out that, in a recent set of papers [39,40], all the three [††] super cohomological operators $\tilde{d}, \tilde{\delta}, \tilde{\Delta}$ have been exploited in the generalized versions of the horizontality condition for the derivation of the nilpotent (anti-)BRST, (anti-)co-BRST and a bosonic symmetry [‡‡] transformations for the two-dimensional (2D) free Abelian gauge theory defined on the four $(2+2)$-dimensional supermanifold.

In a very recent paper [41], the analogue of the Hodge duality $*$ operation has been defined on the six $(4+2)$-dimensional supermanifold by exploiting the mathematical power of $\tilde{d}, \tilde{\delta}$. Furthermore, the non-local, non-covariant and nilpotent dual(co)-BRST symmetry transformations for the gauge- and the (anti-)ghost fields have been obtained in the framework of six $(4+2)$-dimensional superfield formulation for the 4D Abelian gauge theory [41].

[††]On an ordinary manifold without a boundary, the operators (d, δ, Δ) form a set which is popularly known as the de Rham cohomological operators of differential geometry. The operators $(\delta)d$ are called as the (co-)exterior derivatives (with $d = dx^\mu \partial_\mu, \delta = \pm * d *, d^2 = \delta^2 = 0, * =$ Hodge duality operation on the manifold) and $\Delta = (d+\delta)^2 = d\delta + \delta d$ is known as the Laplacian operator (with $[\Delta, d] = [\Delta, \delta] = 0$) [11-15].

[‡‡]This symmetry s_w (with $s_w^2 \neq 0$) is equal to the anticommutator (i.e. $s_w = \{s_b, s_d\} = \{s_{ab}, s_{ad}\}$) of the nilpotent ($s_{(a)b}^2 = s_{(a)d}^2 = 0$) (anti-)BRST $s_{(a)b}$ and (anti-)co-BRST $s_{(a)d}$ transformations [39,40].

However, the non-local, non-covariant and nilpotent transformations for the *matter* (Dirac) fields, that exist in literature (see, e.g., [42,43] for details), have not yet been derived for the interacting Abelian gauge theory in superfield formulation (see, e.g., [41] for details). It would be a very nice endeavour to obtain these nilpotent symmetries for the matter fields by exploiting the invariance of the *non-local* (super)currents on the six-dimensional (super)manifolds in the framework of the augmented superfield formalism. Such kind of nilpotent symmetries for the matter fields also exist for the interacting 4D non-Abelian gauge theory [43]. It would be a very challenging endeavour to capture these nilpotent symmetries in the framework of the six $(4+2)$-dimensional superfield formalism for the 4D interacting non-Abelian gauge theory in the general scheme of the augmented superfield formulation. These are some of the issues that are under investigation at the moment.

References

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[2] For a review, see, e.g., K. Sundermeyer, *Constrained Dynamics: Lecture Notes in Physics* vol. 169 (Berlin, Springer-Verlag, 1982).

[3] See, e.g., K. Huang, *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1982).

[4] See, e.g., I. A. Batalin and I. V. Tyutin, *Int. J. Mod. Phys.* **A6**, 3255 (1991).

[5] A. S. Schwarz, *Lett. Math. Phys.* **2**, 247 (1978).

[6] E. Witten, *Commun. Math. Phys.* **117**, 353 (1988); *ibid.* **121**, 351 (1989).

[7] For an extensive review, see, e.g., D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* **209**, 129 (1991)

[8] See, e.g., R. Gopakumar and C. Vafa, "M-theory and Topological Strings—I and II", hep-th/9808187 and hep-th/9812127.

[9] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* vols. 1 and 2 (Cambridge University Press, Cambridge, 1987).

[10] J. Polchinski, *String Theory* vols. 1 and 2 (Cambridge University Press, Cambridge, 1998).

[11] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980).

[12] S. Mukhi S and N. Mukunda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern, New Delhi, 1990).

[13] J. W. van Holten, *Phys. Rev. Lett.* **64**, 2863 (1990); *Nucl. Phys.* **B339**, 158 (1990).

[14] K. Nishijima, *Prog. Theor. Phys.* **80**, 897 (1988); *ibid.* **80**, 905 (1988).

[15] G. Fülöp and R. Marnelius, *Nucl. Phys.* **B456**, 422 (1985); H. Aratyn, *J. Math. Phys.* **31**, 1240 (1990).

[16] See, e.g., E. Witten, *Mod. Phys. Lett.* **A5**, 487 (1990).

[17] See, e.g., J. Gomis, J. Paris and S. Samuel, *Phys. Rep.* **259**, 1 (1995).

[18] J. Thierry-Mieg, *J. Math. Phys.* **21**, 2834 (1980); *Nuovo Cimento* **56A**, 396 (1980).

[19] M. Quiros, F. J. De Urries, J. Hoyos, M. L. Mazon and E. Rodrigues, *J. Math. Phys.* **22**, 1767 (1981).

[20] L. Bonora and M. Tonin, *Phys. Lett.* **B98**, 48 (1981); L. Bonora, P. Pasti and M. Tonin, *Nuovo Cimento* **63A**, 353 (1981).

[21] R. Delbourgo and P. D. Jarvis, *J. Phys. A: Math Gen* **15**, 611 (1981); R. Delbourgo, P. D. Jarvis and G. Thompson, *Phys. Lett.* **B109**, 25 (1982).

[22] L. Baulieu and J. Thierry-Mieg, *Nucl. Phys.* **B197**, 477 (1982); *ibid.* **B228**, 259 (1982); L. Alvarez-Gaumé and L. Baulieu, *ibid.* **B212**, 255 (1983).

[23] D. S. Hwang and C. -Y. Lee, *J. Math. Phys.* **38**, 30 (1997).

[24] P. M. Lavrov, P. Yu. Moshin and A. A. Reshetnyak, *Mod. Phys. Lett.* **A10**, 2687 1995; P. M. Lavrov and P. Yu. Moshin, *Phys. Lett.* **B508**, 211 (2001).

[25] N. R. F. Braga and A. Das, *Nucl. Phys.* **B442**, 655 (1995).

[26] S. Weinberg, *The Quantum Theory of Fields: Modern Applications* vol. 2 (Cambridge University Press, Cambridge, 1996).

[27] N. Nakanishi and I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific, Singapore, 1990).

[28] R. P. Malik, *Phys. Lett.* **B584**, 210 (2004), hep-th/0311001.

[29] R. P. Malik, *J. Phys. A: Math. Gen.* **37**, 5261 (2004), hep-th/0311193.

[30] R. P. Malik, *Mod. Phys. Lett.* **A20**, 1767 (2005), hep-th/0402123.

[31] R. P. Malik, *Int. J. Geom. Meth. Mod. Phys.* **1**, 467 (2004), hep-th/0403230.

[32] R. P. Malik, *Ann. Phys.* (N. Y.) **307**, 1 (2003), hep-th/0205135.

[33] R. P. Malik, *J. Phys. A: Math. Gen.* **35**, 8817 (2002), hep-th/0204015.

[34] K. Nishijima, *Progress in Quantum Field Theory* eds. H. Ezawa and S. Kamefuchi, (North-Holland, Amsterdam, 1986); For an extensive review, see, e.g., K. Nishijima, *Czech. J. Phys.* **46**, 1 (1996).

[35] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).

[36] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin, 1990).

[37] See, e.g., I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics: A Practical Introduction* (Adam Hilger, Bristol, 1982).

[38] G. Curci and R. Ferrari, *Phys. Lett.* **B63**, 51 (1976); *Nuovo Cimento* **32A**, 151 (1976).

[39] R. P. Malik, *Phys. Lett.* **B521**, 409 (2001), hep-th/0108105.

[40] R. P. Malik, *J. Phys. A: Math. Gen.* **35**, 3711 (2002), hep-th/0106215.

[41] R. P. Malik, *J. Phys. A: Math. Gen.* **37**, 1059 (2003), hep-th/0306182.

[42] See, e.g., M. Lavelle and D. McMullan, *Phys. Rev. Lett.* **71**, 3758 (1993).

[43] V. O. Rivelles, *Phys. Rev.* **D53**, 3247 (1986).